Effect of a non null pressure on the evolution of perturbations in the matter dominated epoch

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ABSTRACT

We analyze the effect of pressure on the evolution of perturbations of an Einstein-de Sitter Universe in the matter dominated epoch assuming an ideal gas equation of state. For the sake of simplicity the temperature is considered uniform. The goal of the paper is to examine the validity of the linear approximation. With this purpose the evolution equations are developed including quadratic terms in the derivatives of the metric perturbations and using coordinate conditions that, in the linear case, reduce to the longitudinal gauge. We obtain the general solution, in the coordinate space, of the evolution equation for the scalar mode, and, in the case of spherical symmetry, we express this solution in terms of unidimensional integrals of the initial conditions: the initial values of the Newtonian potential and its first time derivative. We find that the contribution of the initial first time derivative, which has been systematically forgotten, allows to form inhomogeneities similar to a cluster of galaxies starting with very small density contrast. Finally, we obtain the first non linear correction to the linearized solution due to the quadratic terms in the evolution equations. Here we find that a non null pressure plays a crucial role in constraining the non linear corrections. It is shown, by means of examples, that reasonable thermal velocities at the present epoch (not bigger than 10^{-6}) make the ratio between the first non linear correction and the linear solution of the order of 10^{-2} for a galaxy cluster inhomogeneity.

Subject headings: cosmology: theory, galaxies: clusters: general

1. Introduction

The relativistic theory of the evolution of perturbations was initiated in 1946 by Lifshitz using a special coordinate condition known as the synchronous gauge. He linearized the Eintein's equations to obtain the evolution of perturbations. In fact, he found plane wave solutions for the radiation dominated epoch, assuming $p = (1/3)\rho$ as equation of state, and for the matter dominated epoch, neglecting the effects of the pressure. The theory, with subsequent improvements, is referenced in many books of Cosmology (Peebles 1980; Zel'dovich & Novikov 1983; Landau & Lifshitz 1979; Weinberg 1972).

However, the synchronous coordinate condition has two great drawbacks. The first one is consequence of the fact that it does not completely fix the coordinate system, allowing the existence of gauge modes. This problem can be handled using a gauge invariant version of the theory, started by Bardeen (1934) and collected by Mukhanov, Feldman & Branderberger (1992). This last review also shows an easier way to obtain gauge invariant equations using a coordinate condition that does not allow the existence of gauge modes. These coordinate conditions define what is known as the longitudinal gauge.

The other inconvenience of the synchronous gauge is that the metric perturbation and the density contrast both depend on the second space-like derivatives of a potential. Then, great values for the density contrast imply great values for the metric perturbation, and in consequence the linear approximation in this gauge fails when the density contrast is bigger than unity. On the contrary, in the longitudinal gauge the metric perturbation is proportional to a potential while the density contrast is proportional to the laplacian of the same potential. So, the metric perturbation can be a very small quantity while the corresponding density contrast can achieve values greater than unity. For example, galaxy clusters develop a potential of the order of $\phi/c^2 \leq 10^{-5}$ varying at scales of $R \approx 1Mpc/6000h$; then, using the relation $\delta = (1/6)\Delta\phi$ for the density contrast, in adimensional coordinates, we get $\delta \approx 1/(6R^2) \approx 10^3$ for $h = 0.5$. This makes, in principle, possible the validity of the linear approximation in the longitudinal gauge to study the formation of inhomogeneities similar to galaxy clusters.

Then, the question arises why the linear approximation (linear in the metric perturbation) is always considered inaccurated to describe the evolution when the density contrast is bigger than unity. In this paper we are interested in analyzing when the linear approximation begins to fail in describing the evolution of such objects. To do that, according to the previous paragraph, it is necessary to take into account here on that the spatial derivatives of the potential can be much bigger than the potential. Let us to point, in advance, that the pressure plays a crucial role in this issue.

So, in section 2 we write Einstein's equations in evolutive form, keeping quadratic terms in the first derivatives of the metric perturbation and neglecting quadratic terms in the potential. We use coordinates which simplify the evolution of the tensor components of the metric perturbation and which reduce, in the linear case, to the longitudinal gauge. Moreover, to complete the evolution equations we need to give the stress tensor. In the matter dominated epoch and after the decoupling with radiation, the temperature of matter $T(t)$ decreases as $1/a^2(t)$, where $a(t)$ is the expansion factor, and the pressure becomes so small that it is usually neglected. But, as we shall see in this paper a non null pressure is necessary to keep valid the linear approximation. Then, we will consider the simple case of an ideal gas with an equation of state of the form $p = (a_o^2 T_o / m a^2(t))\rho$. Notice that, although the temperature is becoming very small, the evolution will increase the energy density and this is the reason for keeping the pressure. Under these conditions we write down, in section 3, the evolution equations of the gravitational potential in the linear approximation.

The evolution equations have two degrees of freedom: the potential and its first time derivative. In section 4 we find the general solution $\phi^{(0)}$ of the linear evolution equations for arbitrary initial conditions. Next, we use this result in section 5 to obtain the evolution of the density contrast and the macroscopic velocity starting from appropriated initial conditions to form an inhomogeneity similar to a galaxy cluster. The characteristic length of the structure will be given by the parameter $\epsilon = \tau(1+z_i)^{1/2}$, as a sort of Jeans length, where z_i is the initial redshift and $\tau = \sqrt{T_o/m}$ represents the present value of the random mean square velocity (r.m.s. velocity).

Finally, we face the problem of the validity of the linear approximation. The validity criterion, working in the longitudinal gauge, cannot be based on the value of the density contrast, as we have commented above. Instead, it should be based on the value of the non linear corrections of the evolution equations. So, in section 6 we estimate the first non linear correction, $\phi^{(1)}$, due to the quadratic terms in the Einstein's equations, and obtain an upper bound estimation for the quotient $\Gamma = |\phi^{(1)}/\phi^{(0)}|$. The linear approximation will be considered suitable if this quotient is small, although the density contrast has reached a great value. With this validity criterion, an inhomogeneity similar to a cluster of galaxies could be described with the linear approximation because we obtain a Γ of the order of 10^{-2}. Notice that, if we take the well known solution for $p = 0$, we obtain Γ bigger than unity. So, the pressure plays a crucial role for the validity of the linear approximation.

2. The evolution equations

We assume in this paper the concepts and notations usual of the $3+1$ formalism of general relativity (Smarr & York 1978). We shall consider a perturbation of an Einstein-de Sitter Universe, so we put the metric in the form:

$$ds^2 = -\alpha^2(\phi)dt^2 + \gamma_{ij}dx^i dx^j \tag{1}$$

The shift vector β_i has been taken null, and the lapse function α will be choosen conveniently later. We write the tridimensional metric γ_{ij} in terms of a scalar ϕ and a trace-less tensor:

$$\gamma_{ij} = a^2((1-2\phi)\delta_{ij} + \sigma_{ij}) \tag{2}$$

where $a = a(t)$ denotes the scale factor of the Einstein-de Sitter Universe, and σ_{ij} is a tridimensional tensor verifying $\delta^{mn}\sigma_{mn} = 0$. In the following we shall neglect quadratic terms in the metric perturbations, ϕ and σ_{ij}, but those which are quadratic in its first derivatives (as was explained in the introduction). We shall need the extrinsic curvature of the surfaces t =constant,

$$K_{ij} := -\frac{1}{2\alpha}\partial_t\gamma_{ij} = -\frac{a^2}{\alpha}\{(H(1-2\phi) - \partial_t\phi)\delta_{ij} + H\sigma_{ij} + \frac{1}{2}\partial_t\sigma_{ij}\},$$

being $H = \dot{a}/a$ the Hubble constant, and the Ricci tensor of the tridimensional metric γ_{ij}:

$$R_{ij} = (1+2\phi)\phi_{,ij} + 3\phi_{,i}\phi_{,j} + ((1+2\phi)\Delta\phi + (\nabla\phi)^2)\delta_{ij} - \frac{1}{2}\Delta\sigma_{ij} - \delta^{mn}\sigma_{(im,mj)}$$

where the operators Δ and ∇ are referred to the euclidean tridimensional metric.

Splitting the energy tensor in parallel and orthogonal components to the vector field $u = (1/\alpha)\partial_t$,

$$T^{\mu\nu} = \rho u^\mu u^\nu + ph^{\mu\nu} + q^\mu u^\nu + q^\nu u^\mu + \pi^{\mu\nu}, \tag{3}$$

one gets the corresponding energy density, flux of energy and stress tensor. Then, a Cauchy problem with constraints can be stated in General Relativity (Bruhat & York 1980). Over a space-like surface $t = t_i$ a tridimensional metric and an extrinsic curvature tensor are suposed to be given. These tensors evolve in time according to the following equations:

$$\begin{aligned} \partial_t\gamma_{ij} &= -2\alpha K_{ij} \\ \partial_t K_{ij} &= -D_iD_j\alpha + \alpha(R_{ij} + trKK_{ij} - 2K_{ia}K^a_j) + 4\pi G\alpha(p-\rho)\gamma_{ij} - 8\pi G\alpha\pi_{ij} \end{aligned} \tag{4}$$

with trK representing the tridimensional metric trace of the extrinsic curvature tensor and D_i the tridimensional covariant derivative. In these equations the components of the stress tensor, p and π_{ij}, must be choosen from the beginning, as we will do below.

The energy density and the flux of energy are linked by constraint conditions to the Ricci tensor and the extrinsic curvature:

$$\begin{aligned} 16\pi G\rho &= (trK)^2 - tr(K^2) + R \\ 8\pi G q_i &= D^a K_{ai} - D_i trK \end{aligned} \tag{5}$$

where R is the scalar curvature of the tridimensional metric. If one knows at some initial instant t_i the flux of energy and the energy density, one must solve first the constraint equations (5) to determine a valid set of initial conditions $\gamma^*_{ij}(x), K^*_{ij}(x)$. Then, the evolution equations (4) determine the $\gamma_{ij}(t,x), K_{ij}(t,x)$ for $t > t_i$. Substituting them into the constraint equations one gets the evolution of the energy density and the flux of energy.

Finally, we shall assume a one-component Universe with a pressure tensor of the form:

$$\pi_{ij} = A[\phi_{,ij}]^t + \pi^{(2)}_{ij}(\phi,t) \tag{6}$$

$$p = p_B + E\Delta\phi + p^{(2)}(\phi,t) \tag{7}$$

where $[\phi_{,ij}]^t$ means trace-less component, A and E are only functions of time and $p^{(2)}$ and $\pi^{(2)}_{ij}$ are a scalar and a second order 3-tensor formed with the 3-vectors $\phi_{,i}$, $\sigma_{im,m}$ and its first time derivatives respectively. This assumption is quite general because it allows to consider an ideal gas as well as solutions of an Einstein-Vlasov problem. Next, we shall develope the evolution equations taking into account these last expressions.

Let us start splitting the second evolution equation into trace-less and trace part equations. The trace-less part is:

$$-\frac{a^2}{2}\partial_t^2\sigma_{ij} - \frac{3}{2}a^2 H\partial_t\sigma_{ij} + \frac{1}{2\alpha}\Delta\sigma_{ij} + \frac{1}{\alpha}\sigma^t_{(im,mj)} = S_{ij} \tag{8}$$

where

$$S_{ij} = (\alpha(1+2\phi) - \alpha' - 8\pi G\alpha A)[\phi_{,ij}]^t + (-\alpha'' - 2\alpha' + 3\alpha)[\phi_{,i}\phi_{,j}]^t - 8\pi G\alpha\pi^{(2)}_{ij}. \tag{9}$$

An appropriate election of the lapse function $\alpha(\phi)$ can simplify the problem. Lifshitz used the Gaussian gauge: $\alpha = 1$, but this choice has more than one inconvenience, as we have pointed out at the introduction. Looking at equation (9) we see that the best choice is to take α such that the coefficient of the Hessian vanishes. That means to take α as the solution of the equation:

$$\alpha(1+2\phi) - \alpha' - 8\pi G\alpha A = 0$$

which is $\alpha = e^{b_1\phi+\phi^2}$, with $b_1 = 1 - 8\pi GA$. With this election, the coefficient of $[\phi_{,i}\phi_{,j}]^t$ in the expression of S_{ij} becomes $-2 + 32\pi GA$, and the evolution of the trace-less component results:

$$\partial_t^2\sigma_{ij} + 3H\partial_t\sigma_{ij} - \frac{1}{a^2}(\Delta\sigma_{ij} + 2\sigma^t_{(im,mj)}) = \frac{4}{a^2}(1 - 16\pi GA)[\phi_{,i}\phi_{,j}]^t + \frac{16\pi G(1-8\pi GA)}{a^2}\pi^{(2)}_{ij}$$

As to the trace component of the evolution equation, it writes down as:

$$\partial_t^2\phi + 4H\partial_t\phi - (\frac{8\pi GA}{3a^2} + 4\pi GE)\Delta\phi + \frac{1}{12a^2}\partial_i\partial_m\sigma_{im} = \frac{1}{2}(\partial_t\phi)^2 - \frac{1}{6a^2}(\nabla\phi)^2 + 4\pi Gp^{(2)}$$

We have found convenient to introduce the conformal time η, defined by $dt = a^2 d\eta$. In this time coordinate the expansion factor writes down as $a(\eta) = a_o\eta^2$ with a_o related to the Hubble constant by $a_o = 2/H_o$. Then,the final form of the evolution equations is:

$$\partial_\eta^2\phi + \frac{6}{\eta}\partial_\eta\phi - \frac{4\pi G}{3}(2A + 3Ea^2)\Delta\phi + \frac{1}{12}\sigma_{im,im} = \frac{1}{2}(\partial_\eta\phi)^2 - \frac{1}{6}(\nabla\phi)^2 + 4\pi Ga^2p^{(2)} \tag{10}$$

$$\partial_\eta^2\sigma_{ij} + \frac{4}{\eta}\partial_\eta\sigma_{ij} - \Delta\sigma_{ij} - 2[\sigma_{(jm,im)}]^t = 4(1 - 16\pi GA)[\phi_{,i}\phi_{,j}]^t + 16\pi G(1 - 8\pi GA)\pi_{ij}^{(2)} \tag{11}$$

In addition to these equations, the constraint conditions should be considered:

$$4\pi G\rho = \frac{1}{a^2}\Delta\phi - \frac{3H}{a}\phi_{,\eta} + \frac{3H^2}{2}(1 - 2b_1\phi) + \frac{5}{2a^2}(\nabla\phi)^2 + \frac{3}{2a^2}(\phi_{,\eta})^2 + \frac{1}{4a^2}\sigma_{im,im} \tag{12}$$

$$4\pi Gq_i = -b_1H\phi_{,i} - \frac{1}{a}\phi_{,\eta i} + \frac{1}{2a}\sigma_{im,m\eta} + H\sigma_{im,m} \tag{13}$$

So, it remains to give the functions A, E, $p^{(2)}$ and $\pi_{ij}^{(2)}$ appearing in the pressure tensor. As we are interested in the matter dominated epoch, it is reasonable to consider the one-component Universe as an ideal gas, with energy tensor $T_{\mu\nu} = \rho_c w_\mu w_\nu + p_c(g_{\mu\nu} + w_\mu w_\nu)$, equation of state $p_c = (T/m)\rho_c$ and temperature evolving as $T = const/a^2$. In the Appendix A we show how this assumption means to take $A = 0$ and $E \approx \tau^2/4\pi Ga^4$, where $\tau = \sqrt{T_o/m}$ is the r.m.s. veloctiy at the present epoch, and neglect the second order expressions $p^{(2)}$ and $\pi_{ij}^{(2)}$.

Let us repoduce the complete set of equations:

$$\partial_\eta^2\phi + \frac{6}{\eta}\partial_\eta\phi - \frac{\tau^2}{\eta^4}\Delta\phi + \frac{1}{12}\sigma_{im,im} = \frac{1}{2}(\partial_\eta\phi)^2 - \frac{1}{6}(\nabla\phi)^2 \tag{14}$$

$$\partial_\eta^2\sigma_{ij} + \frac{4}{\eta}\partial_\eta\sigma_{ij} - \Delta\sigma_{ij} - 2[\sigma_{(jm,im)}]^t = 4[\phi_{,i}\phi_{,j}]^t \tag{15}$$

$$\delta = \frac{\eta^2}{6}\Delta\phi - \eta\phi_{,\eta} - 2\phi + \frac{\eta^2}{24}\sigma_{im,im} \tag{16}$$

$$4\pi Gq_i = -H\phi_{,i} - \frac{1}{a}\phi_{,\eta i} + \frac{1}{2a}\sigma_{im,m\eta} + H\sigma_{im,m} \tag{17}$$

where we have substituted the energy density ρ by the density contrast δ using the relation $\delta = (\rho - \rho_{_B})/\rho_{_B}$, with $\rho_{_B} = 3H^2/8\pi G$ the background energy density.

2.1. The longitudinal gauge

The longitudinal gauge, unlike the Gaussian gauge, fixes definitely the coordinates. This is a well known fact, but let us give here an argument, which may be useful for other purposes.

If we start with the Robertson-Walker (R-W) metric in canonic coordinates, $ds^2 = -d\bar{t}^2 + a^2(\bar{t})\delta_{ij}d\bar{x}^i d\bar{x}^j$ and we introduce new coordinates:

$$\begin{aligned}\bar{t} &= t + \varphi(t, x)\\ \bar{x}^i &= x^i + \xi_i(t, x)\end{aligned}$$

and impose the coordinate conditions $g_{oi} = 0$, one puts the metric in the form:

$$ds^2 = -(1 + 2\dot{\varphi})dt^2 + a^2(t)[(1 + 2H\varphi)\delta_{mn}dx^m dx^n + 2\xi_{(m,n)}dx^m dx^n] \tag{18}$$

with

$$\xi_m(t, x) = \frac{1}{2}\psi_{,m} + \zeta_m(x), \qquad \psi = 2\int \frac{1}{a^2}\varphi dt$$

being $\dot{\varphi}$ the time derivative of φ, and ζ_m arbitrary functions of the space-like coordinates. Then, the lapse function and the tridimensional metric are:

$$\begin{aligned}\alpha &= 1 + \dot{\varphi}\\ \gamma_{ij}^{RW} &= a^2(t)\left[(1 + 2H\varphi + \frac{1}{3}\Delta\psi + \frac{2}{3}\zeta_{m,m})\delta_{ij} + \psi^t_{,ij} + 2\zeta^t_{(m,n)}\right]\end{aligned}$$

Comparing these expressions with equation (2) we obtain the gravitational potential and the trace-less tensor as:

$$\begin{aligned}-2\phi &= 2H\varphi + \frac{1}{3}\Delta\psi + \frac{2}{3}\zeta_{m,m}\\ \sigma_{mn} &= \psi^t_{,mn} + 2\zeta^t_{(m,n)}\end{aligned}$$

Consequently, if we choose the longitudinal gauge, i.e. $\alpha = 1 + \phi$, the function φ, which define the new time coordinate, should satisfy the equation:

$$\partial_t^2\varphi + \frac{1}{3a^2}\Delta\varphi + \dot{H}\varphi + H\partial_t\varphi = 0. \tag{19}$$

Notice that the coefficients of the second time derivatives and of the laplacian operator in this equation have the same sign. This makes impossible to construct a time-like foliation with the function φ, apart from the case $\varphi = 0$. So, we can conclude that if we develope a R-W perturbation in the longitudinal gauge, it is impossible to recover a R-W space-time in other coordinates.

However, if one chooses the Gaussian gauge, i.e. $\alpha = 1$, the function φ should now satisfy the equation $\dot{\varphi} = 0$, making possible to introduce new time coordinates. This forces to characterize the gauge modes, i.e. the false metric perturbations.

3. The linear evolution equations

We shall study now the linear equations:

$$\partial_\eta^2\phi + \frac{6}{\eta}\partial_\eta\phi - \frac{\tau^2}{\eta^4}\Delta\phi + \frac{1}{12}\sigma_{im,im} = 0 \tag{20}$$

$$\partial_\eta^2\sigma_{ij} + \frac{4}{\eta}\partial_\eta\sigma_{ij} - \Delta\sigma_{ij} - 2[\sigma_{(jm,im)}]^t = 0 \tag{21}$$

$$\delta = \frac{\eta^2}{6}\Delta\phi - \eta\phi_{,\eta} - 2\phi + \frac{\eta^2}{24}\sigma_{im,im} \tag{22}$$

$$4\pi G q_i = -H\phi_{,i} - \frac{1}{a}\phi_{,\eta i} + \frac{1}{2a}\sigma_{im,m\eta} + H\sigma_{im,m} \tag{23}$$

Firstly, we consider that the trace-less symmetric tensor σ_{ij} can be decomposed (York 1973) in a transverse part σ_{ij}^T, verifying $\sigma_{im,m}^T = 0$, and a longitudinal part σ_{ij}^L, with $\sigma_{im,m}^L = \sigma_{im,m}$. These two components are orthogonal and evolve independently. Then, one can distinguish three modes in our problem: the scalar one ϕ, the transverse tensor σ_{ij}^T and the longitudinal tensor σ_{ij}^L. The constraint equations show that the scalar mode is the most important contribution to the density contrast (22) and to a rotational-free flux of matter (23); the longitudinal tensor mode contributes weakly to the density contrast and, what is more interesting, it is the only possibility of producing a non null rotational component of the velocity field. Relative to the tensor transverse mode, it does not contribute nor to the density neither to the energy flux, in fact it represents the emission of gravitational waves.

Next, given that the scalar and the longitudinal modes are coupled and that the double divergence of the longitudinal part appears in the evolution of ϕ, we can rewrite the equations introducing the scalar $\theta = -(1/12)\sigma_{im,im}$. In this manner, the evolution of σ_{ij} gives the evolution of θ and the weak coupling between the scalar mode ϕ and the longitudinal one θ is given by:

$$\partial_\eta^2\phi + \frac{6}{\eta}\partial_\eta\phi - \frac{\tau^2}{\eta^4}\Delta\phi = \theta \tag{24}$$

$$\partial_\eta^2\theta + \frac{4}{\eta}\partial_\eta\theta - \frac{7}{3}\Delta\theta = 0 \tag{25}$$

Both equations are hyperbolic, but there exist a great difference between them due to the time dependent coefficient η^{-4} that appears in the evolution of ϕ. This coefficient causes that the characteristic curves of the first equation do not scape to infinity as do the charateristics of θ, because in this case the laplacian operator has a constant coefficient. This can be seen clearly in the case of spherical symmetry, where the charateristic curves $r(\eta)$ of (24) tends to a finit limit as the conformal time tends to infinity; while for the variable θ, the

characteristic curves of (25) scape to infinity. This makes possible that a linear hyperbolic equation might describe the increasing of density in bounded regions.

Morever, this difference makes the coupling between the scalar and longitudinal modes almost irrelevant, because small initial values for θ in a small region disperse to infinity.

Finally, we observe an important difference between our evolution equations an those in the Gaussian gauge (lapse function $\alpha = 1$ and shift vector $\beta = 0$) used by Lifshitz. In our gauge, the evolution equation for the scalar mode can be reduced to a unique equation for a unique function, while in the Gaussian gauge the scalar mode is described by two coupled equations.

In the next section we shall find the general solution in the coordinate space of the evolution equation for the scalar mode, neglecting the coupling with θ or assuming $\theta(\eta_i, x) = \partial_\eta \theta(\eta_i, x) = 0$.

4. The Cauchy problem of the linear evolution equations

From equation (24) and in the case of null initial conditions for θ, we can consider the following initial value problem:

$$\begin{aligned} \partial_\eta^2 \phi + \frac{6}{\eta}\partial_\eta \phi - \frac{\tau^2}{\eta^4}\Delta\phi &= 0 \\ \phi(\eta_i, x) = \phi_i(x)\ ,\ \partial_\eta \phi(\eta_i, x) &= \phi_i'(x) \end{aligned} \tag{26}$$

with $\phi_i(x)$ and $\phi_i'(x)$ two arbitrary functions. We shall solve this problem using the method of Fourier transforms.

Firstly, let us denote by $\hat{\phi}(\eta, s)$, with $s \in \mathbb{R}^3$, the Fourier transform of $\phi(\eta, x)$ with respect to the spatial coordinates. In the Fourier space, equation (26) transforms into an initial value problem for an ordinary differential equation:

$$\begin{aligned} \partial_\eta^2 \hat{\phi} + \frac{6}{\eta}\partial_\eta \hat{\phi} + \frac{\tau^2}{\eta^4} s\cdot s\hat{\phi} &= 0 \\ \hat{\phi}(\eta_i, s) = \hat{\phi}_i(s)\ \ \partial_\eta \hat{\phi}(\eta_i, s) &= \hat{\phi}_i'(s) \end{aligned} \tag{27}$$

So, the first task is that of constructing a system of fundamental solutions, which consists on two solutions $\hat{\phi}_1(\eta, s)$, $\hat{\phi}_2(\eta, s)$ that satisfy the initial conditions:

$$\begin{aligned} \hat{\phi}_1(\eta_i, s) = 1 \qquad &, \qquad \partial_\eta \hat{\phi}_1(\eta_i, s) = 0 \\ \hat{\phi}_2(\eta_i, s) = 0 \qquad &, \qquad \partial_\eta \hat{\phi}_2(\eta_i, s) = 1 \end{aligned}$$

These fundamental solutions can be obtained using complex Laplace transforms (Smirnov), having the following result:

$$\hat{\phi}_1(\eta, s) = \frac{3}{\epsilon^3}\left(\frac{\sin kg}{k^3} - g\frac{\cos kg}{k^2}\right) + \frac{\eta_i(3\eta - \eta_i)}{\epsilon\eta^2}\frac{\sin kg}{k} + \frac{\eta_i^2}{\eta^2}\cos kg \tag{28}$$

$$\begin{aligned}\hat{\phi}_2(\eta, s) &= \frac{9\eta_i}{\epsilon^5}\left(\frac{\sin kg}{k^5} - g\frac{\cos kg}{k^4} - g^2\frac{\sin kg}{3k^3}\right) + \\ &+ \frac{3\eta_i^2}{\epsilon^3\eta}\left(\frac{\sin kg}{k^3} - g\frac{\cos kg}{k^2}\right) + \frac{\eta_i^3}{\epsilon\eta^2}\frac{\sin kg}{k}\end{aligned} \tag{29}$$

where k stands for the modulus of s, $k = \sqrt{s \cdot s}$, $\epsilon = \tau/\eta_i$, and $g = \epsilon(1 - \frac{\eta_i}{\eta})$. Then, the solution of (27), in the Fourier space, is expressed in terms of the fundamental system as:

$$\hat{\phi}(\eta, s) = \hat{\phi}_i(s)\hat{\phi}_1(\eta, s) + \hat{\phi}'_i(s)\hat{\phi}_2(\eta, s). \tag{30}$$

The next task is to obtain the Green's functions $Q_1(\eta, x)$ and $Q_2(\eta, x)$, defined as the inverse Fourier transform of the fundamental system $\{\hat{\phi}_1(\eta, s),\ \hat{\phi}_2(\eta, s)\}$. As we show in Appendix B, these Green functions are:

$$Q_1(\eta, x) = \frac{3}{4\pi\epsilon^3}H(g - r) + (\frac{3\eta_i}{\epsilon\eta} - \frac{\eta_i^2}{\epsilon\eta^2})\frac{\delta_D(r - g)}{4\pi g} + \frac{\eta_i^2}{4\pi\eta^2}\partial_g\left(\frac{\delta_D(r - g)}{g}\right) \tag{31}$$

$$Q_2(\eta, x) = \left(\frac{3\eta_i}{8\pi\epsilon^5}(g^2 - r^2) + \frac{3\eta_i^2}{4\pi\eta\epsilon^3}\right)H(g - r) + \frac{\eta_i^3}{4\pi\epsilon\eta^2}\frac{\delta_D(r - g)}{g} \tag{32}$$

Therefore, the solution of (26) in the coordinate space is expressed as the convolution product of the Green's functions with the initial conditions:

$$\phi(\eta, x) = Q_1(\eta, x) * \phi_i(x) + Q_2(\eta, x) * \phi'_i(x) \tag{33}$$

where $*$ stands for the convolution product with respect to the spatial coordinates. Looking at the Green's functions we can observe that the solution tends rapidly to a limit when the conformal time tends to infinity, this limit has a simple expression:

$$\phi(\infty, x) = \frac{3}{4\pi\epsilon^3}\int_{|x-\xi|<\epsilon}\phi_i(\xi)d\xi + \frac{3\eta_i}{8\pi\epsilon^5}\int_{|x-\xi|<\epsilon}(\epsilon^2 - \mid x - \xi \mid^2)\phi'_i(\xi)d\xi \tag{34}$$

as tridimensional integrals of the initial conditions. The ϵ parameter in expression (34) can be also written as $\epsilon = \tau\sqrt{1 + z_i}$, with τ the r.m.s. velocity of the matter component at the present epoch and z_i the initial redshift (recall the relation $1 + z = 1/\eta^2$ in an Einstein-de Sitter Universe). This parameter will be crucial to the study of evolution because it will fix the characteristic length of the evolved structures as a sort of Jeans length.

In the case of spherical symmetry the convolutions reduce to unidimensional integrals, whose expressions are obtained in the Appendix B. They will be used in the sequel to discuss some examples.

5. The evolution of fluctuations with non null thermal motions

Having got the general solution of the linear initial value problem, the constraint equations (22) and (23) determine the density contrast and the flux of matter as simple functionals of the metric perturbations. In the linear approximation we have that these equations for δ and q_m reduce to:

$$\delta(\eta, x) = \frac{\eta^2}{6}\Delta\phi - 2\phi - \eta\phi' \tag{35}$$

$$4\pi G a q_m = -aH\phi_{,m} - \phi_{,\eta m} \tag{36}$$

where we only have to substitute the expression (33) of the solution ϕ. Notice that in the case of an ideal gas, the flux of energy in the longitudinal gauge represents the macroscopic velocity: $q_m = \rho V_m$. In particular we are interested in the velocity norm, whose expression is:

$$\mid V(\eta, x) \mid_\gamma = \frac{1}{1+\delta} \mid \nabla(\frac{\eta}{3}\phi + \frac{\eta^2}{6}\phi_{,\eta}) \mid . \tag{37}$$

Let us study these expressions when the general solution (33) is substituted. We shall assume that at some initial redshift z_i we know the initial conditions for the potential $\phi_i(x)$ and its first time derivative $\phi_i'(x)$. We shall begin with a qualitative description of the evolution of the density contrast based on the following reduced expression for δ:

$$\delta(\eta, x) \approx \frac{\eta^2}{6}(Q_1(\infty, x) * \Delta\phi_i(x) + Q_2(\infty, x) * \phi_i'(x)) \tag{38}$$

$$Q_1(\infty, x) = \frac{3}{4\pi\epsilon^3}H(\epsilon - r) \tag{39}$$

$$Q_2(\infty, x) = \frac{3\eta_i}{8\pi\epsilon^5}(\epsilon^2 - r^2)H(\epsilon - r) \tag{40}$$

where we have only considered the laplacian term neglecting the ϕ and ϕ' contributions and we have also taken the asymptotic values for the Green's functions. The idea is to obtain L^p estimations of the convolutions using the Hőlder inequalities (Hőrmander 1989). In this case we have enough with the relations $\parallel f * g \parallel_\infty \leq \parallel f \parallel_1 \parallel g \parallel_\infty$, where $\parallel f \parallel_\infty$ means $sup \mid f \mid$, and $\parallel f \parallel_1$ means $\int \mid f \mid dx$. We obtain in this way, for $\eta >> \eta_i$, two upper bounds for δ:

$$\parallel \delta(\eta, x) \parallel_\infty \leq \frac{\eta^2}{6}\Big(\parallel Q_1(\infty, x) \parallel_\infty \ \parallel \Delta\phi_i(x) \parallel_1 + \parallel Q_2(\infty, x) \parallel_\infty \ \parallel \Delta\phi_i'(x) \parallel_1 \Big) \tag{41}$$

$$\parallel \delta(\eta, x) \parallel_\infty \leq \frac{\eta^2}{6}\Big(\parallel Q_1(\infty, x) \parallel_1 \ \parallel \Delta\phi_i(x) \parallel_\infty + \parallel Q_2(\infty, x) \parallel_1 \ \parallel \Delta\phi_i'(x) \parallel_\infty \Big) \tag{42}$$

This will allow to reach the main conclusions with little calculations, treating separately each one of our two degrees of freedom $\phi_i(x)$ and $\phi_i'(x)$. In the subsections we shall give more

details in numerical examples where we shall assume spherical symmetry. To do it we need to fix the parameters of the problem. We have three parameters related by the condition $\epsilon = \tau\sqrt{1+z_i}$. As we have mentioned above the final characteristic length of a structure will depend on the value of ϵ, so if we want to discuss galaxy clusters it will be convenient to take $\epsilon \approx 1Mpc/a_o$, $a_o = 6000h^{-1}Mpc$, which is of the order of an Abel's radius for this value of ϵ. And assuming $z_i = 5000$ we obtain $\tau \approx 10^{-6}$ for the r.m.s. velocity at the present epoch. Notice that this value can be supported by hot particles as neutrinos with non null mass.

In these examples, to identify the final structure as something similar to a galaxy cluster, we shall estimate the total mass at the present epoch $\eta = 1$ and inside an Abel's radius $r_a = 1.5h^{-1}Mpc$ as a function of the amplitude A and the initial characteristic length R:

$$M(< r_a) = 4\pi 2.7 \times 10^{11} M_{\odot} \int_0^{r_a} \delta(r, 1, R, A) dr \tag{43}$$

Recall that a typical galaxy cluster has $M(< r_a) \approx 3h^{-1} \times 10^{14} M_{\odot}$.

In the subsections we examine the possibility of generating structures similar to a galaxy cluster starting from reasonable initial conditions for the potential, that is, such that the initial density contrast and the macroscopic velocity be small. We shall also require that in both examples the gravitational potential at the decoupling of matter and radiation be smaller than 6×10^{-5}.

5.1. Initial conditions of the form $\phi_i \neq 0$, $\phi_i' = 0$

Let us start studying the case of $\phi_i \neq 0$ and $\phi_i' = 0$. In this case, from equations (35) and (37), one gets the initial density contrast and the initial macroscopic velocity as: $\delta_i(x) \approx \frac{\eta_i^2}{6}\Delta\phi_i(x)$ and $\mid V(\eta_i, x) \mid_\gamma \approx \frac{\eta_i}{3} \mid \nabla\phi_i \mid$. Let us consider an initial potential of the form $\phi_i(x) = -A(1+r^2/R^2)^{-1/2}$, in which we have two parameters, the amplitude A and the characteristic length R. The laplacian of this function is $\Delta\phi_i = 6B(1+r^2/R^2)^{-5/2}$, where B is given by the relation $A = 2BR^2$ and is proportional to the initial central density contrast, $\delta_i(0) = \eta_i^2 B$. Under these considerations, from equations (41) and (42) we have that the final density contrast is upper bounded by:

$$\parallel \delta(\eta, x) \parallel_\infty \leq minor \;\; of \;\; \left[\frac{\eta^2}{\eta_i^2}\frac{R^3}{\epsilon^3}\delta_i(0) \; , \; \frac{\eta^2}{\eta_i^2}\delta_i(0)\right]$$

So, one distisguishes two cases:

- If $R < \epsilon$ we have $\parallel \delta(\eta, x) \parallel \leq (\eta/\eta_i)^2(R/\epsilon)^3\delta_i(0)$. In this case the density contrast decreases until the instant η_*, for which $(\eta_*/\eta_i)^2(R/\epsilon)^3 = 1$, and then begin to increase

with a power law. This possibility was stated first by Gilbert (Gilbert 1966) using a Newtonian approximation to the Einstein equations.

- If $R > \epsilon$ the object grows from the very beginning, tending to a power law: $\| \delta(\eta, x) \| \leq (\eta/\eta_i)^2 \delta_i(0)$.

Getting on with the example we will assign numerical values to the free parameters A and R. We shall choose them such that the total mass inside an Abel sphere of radius $r_a = 1.5h^{-1}Mpc$ be about $3h^{-1} \times 10^{14}$ solar masses and, at the same time, keeping bounded the gravitational potential by $| \phi(\eta, x) | < 6 \times 10^{-5}$. For example, taking $A = 2.8 \times 10^{-5}$ we can obtain the mass $M(< r_a)$ as a function of the characteristic length R, whose graph is given in Figure 1.

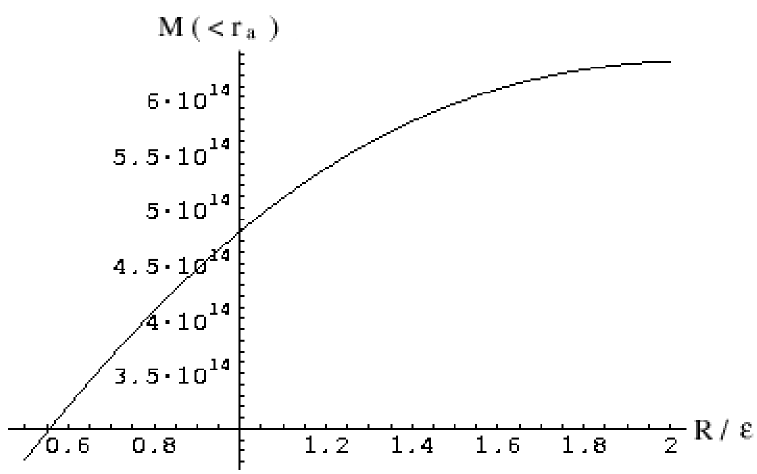

Fig. 1.— The mass $M(< r_a)$ inside an Abel's radius ($r_a = 1.5h^{-1}Mpc$) at the preset epoch as a function of the initial characteristic length R, given by equation (43) when we evolve the density contrast taking into account only the first degree of freedom, i.e., having as initial conditions $\phi_i = -A(1 + r^2/R^2)^{-1/2}$ and $\phi_i' = 0$, with $A = 2.8 \times 10^{-5}$. We assume the values $\epsilon = 1Mpc/a_o$, $a_o = 6000h^{-1}Mpc$, $h = 0.5$ and $z_i = 5000$ for the ϵ-parameter and the initial redshift. The r.m.s. velocity at the present epoch parameter takes the value $\tau = 1.2 \times 10^{-6}$.

This figure shows that we can choose $R = 1.6\epsilon$ to obtain the mass of a typical galaxy cluster. With this election, equations (37) and (38) give the evolution of the velocity and the density contrast, whose graphs at the initial and final time are shown in Figure 2.

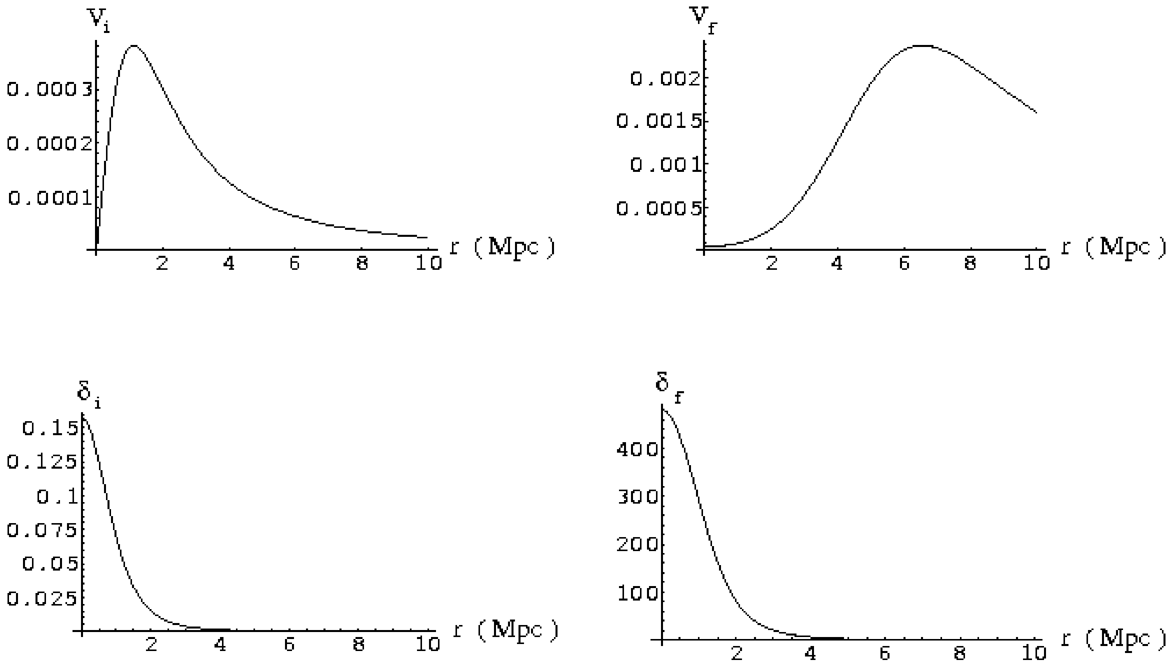

Fig. 2.— Initial (left figures) and present (right figures) values for the density contrast and the macroscopic velocity, evolving the first degree of freedom under the same conditions as in Figure 1 and with $R = 1.6\epsilon$. The mass inside an Abel's radius $r_a = 1.5h^{-1}Mpc$ at the present epoch is about $6 \times 10^{14} M_{\odot}$.

In Figure 3 we show the evolution of the central value of the density contrast in the case where $R < \epsilon$ to illustrate the bouncing of inhomogeneities with typical length R smaller than ϵ, as we have described above.

As conclusion of this subsection we can say that the evolution of Einstein's equations with initial conditions of the form $\{\phi_i(x) \neq 0, \phi_i'(x) = 0\}$ is equivalent to the evolution goberned by Newton's equations (Gilbert 1966) with initial conditions $\{\delta_i(x), V_i(x)\}$ given at the beginning of the subsection. The second example will lead us to a quite different conclusion.

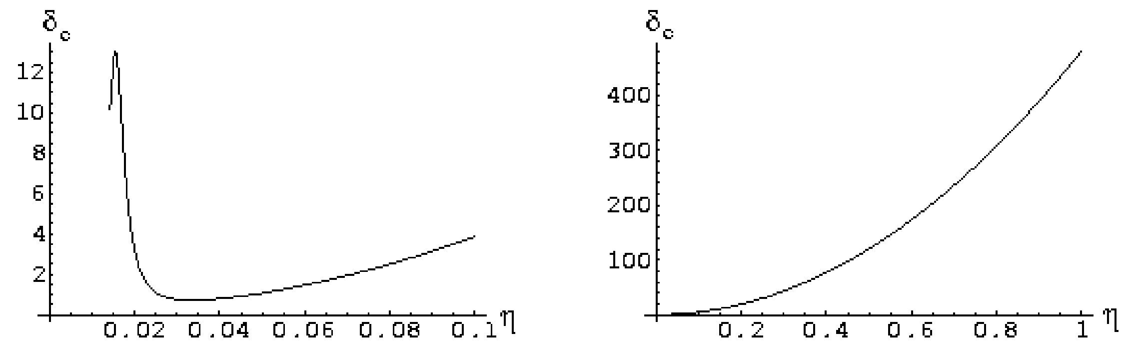

Fig. 3.— The central density contrast obtained evolving the first degree of freedom. The right figure shows the evolution under the same conditions as in Figure 1 and with $R = 1.6\epsilon$. The left figure shows the bouncing produced on the central density contrast when we take an initial characteristic length $R = 0.2\epsilon$. Then we can see that the ϵ parameter plays the role of a Jeans length.

5.2. Initial conditions of the form $\phi_i = 0$, $\phi_i' \neq 0$

Next we are going to study the case where $\phi_i = 0$ and $\phi_i' \neq 0$. In this case, the initial density contrast and the macroscopic velocity are not given by the initial potential but by its initial first time derivative in the form: $\delta_i(x) = -\eta_i \phi_i'$ and $\mid V(\eta_i, x) \mid_\gamma = \frac{\eta_i^2}{6} \mid \nabla \phi_i' \mid$. Now, we shall consider an initial condition of the form $\phi_i'(x) = -A(1 + r^2/R^2)^{-1/2}$, with A and R two free parameters. Defining $A = 2BR^2$ we can write $\Delta\phi_i' = 6B(1 + r^2/R^2)^{-5/2}$ having that the final density contrast is upper bounded by:

$$\parallel \delta(\eta, x) \parallel_\infty \leq minor \;\; of \quad \left[\frac{1}{2}\eta^2 \eta_i \frac{R^3}{\epsilon^3} B \; , \; \frac{1}{5}\eta^2 \eta_i B \right].$$

Let us remark the main difference with the previous case. If we consider here $R \geq \epsilon$, the final density contrast will be bounded by $(1/5)\eta_i B$, but now B is not constrained to be small because the initial density contrast and the macroscopic velocity as functions of B and R are given by:

$$\delta_i(r) = \frac{2\eta_i B R^2}{(1 + r^2/R^2)^{1/2}} \tag{44}$$

$$\mid V(\eta_i, r) \mid_\gamma = \frac{\eta_i^2 B r}{3(1 + r^2/R^2)^{3/2}} \tag{45}$$

having that for objects much smaller than the horizon at the present epoch ($R = 1Mpc/a_o$ with $a_o = 6000h^{-1}Mpc$ implies $R \sim 10^{-4}$), $\eta_i B R^2$ can be small even when $B >> 1$. There-

fore, the second degree of freedom allows to reach great values of the density contrast starting with very small initial density contrast.

As in the previous subsection, we have to take values for A and R such that the total mass inside an Abel sphere of radius $r_a = 1.5h^{-1}Mpc$ be of the order of $3h^{-1} \times 10^{14}$ solar masses and, at the same time, keeping bounded the gravitational potential by $\mid \phi(\eta, x) \mid < 6 \times 10^{-5}$. For example, taking $A = 3 \times 10^{-2}$ we can determine $M(< r_a)$ as a function of R, whose graph is represented in Figure 4.

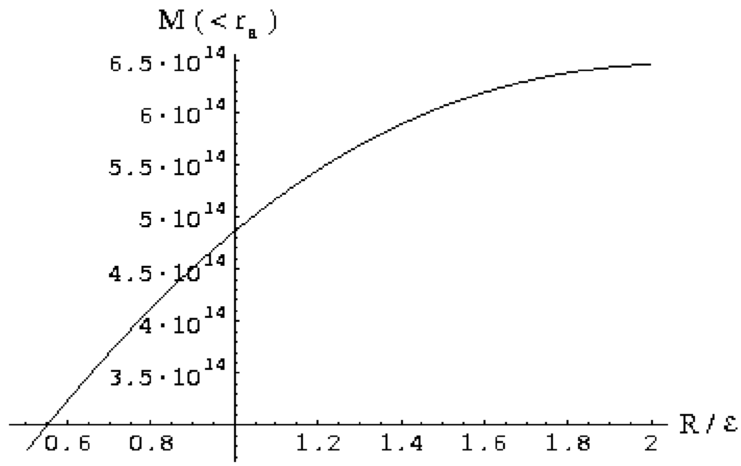

Fig. 4.— The mass $M(< r_a)$ inside an Abel's radius ($r_a = 1.5h^{-1}Mpc$) at the preset epoch as a function of the initial characteristic length R, given by equation (43) when we evolve density contrast taking into account only the second degree of freedom, i.e., having as initial conditions $\phi_i = 0$ and $\phi_i' = -A(1 + r^2/R^2)^{-1/2}$, with $A = 10^{-2}$. We assume the same values for ϵ, z_i, τ and h as in Figure 1.

From this picture we obtain that a good value for the characteristic length is $R = 1.6\epsilon$. These values of the parameters allow us to evolve the density contrast and the velocity, whose evolution is represented in Figure 5.

Then, we can conclude that the second degree of freedom, taking an appropriate value for ϕ_i', allows the formation of great structures starting from very small initial values for the density contrast. This case has no Newtonian analogue because now we have significant initial time derivatives of the gravitational potential. In other words, a Newtonian evolution starting with initial density contrast and macroscopic velocities as given by expressions (44)

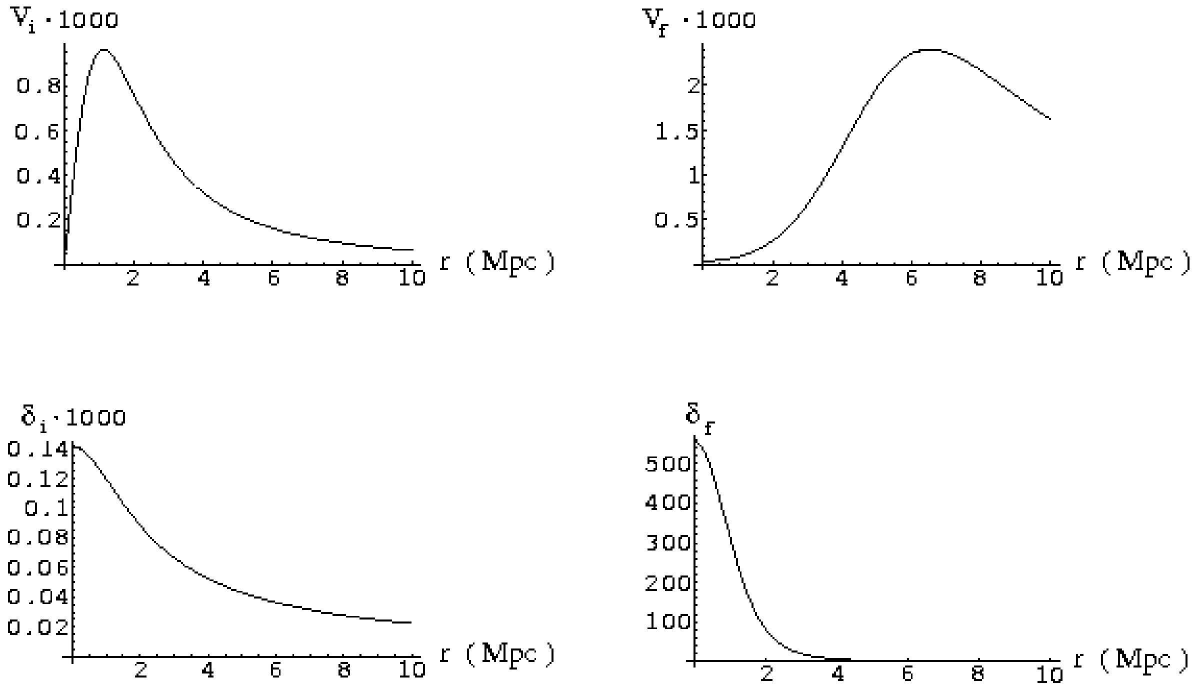

Fig. 5.— Initial (left figures) and present (right figures) values for the density contrast and the macroscopic velocity, evolving the second degree of freedom with the choice of parameters done in Figure 4 and with $R = 1.6\epsilon$. The mass inside an Abel's radius $r_a = 1.5h^{-1}Mpc$ at the present epoch is $6 \times 10^{14} M_\odot$.

and (45) will produce a structure with a very small final density contrast at the present epoch.

Finally, let us remark that the second degree of freedom can be described geometrically as follows: the initial surface $\eta = \eta_i$ has null intrinsic curvature (null laplacian of ϕ_i) and highly inhomogeneous extrinsic curvature (great space derivatives of ϕ_i').

6. On the validity of the linear approximation

In this section we come back to the non linear equations (14) and (15) in order to study the validity of the linear approximation. Introducing the function θ as in section 3, we obtain

a coupled system of evolution equations for the couple of functions (ϕ, θ):

$$\begin{aligned}
\partial_\eta^2 \phi + \frac{6}{\eta}\partial_\eta \phi - \frac{\tau^2}{\eta^4}\Delta\phi &= \theta + \frac{1}{2}(\partial_\eta \phi)^2 - \frac{1}{6}(\nabla\phi)^2 \\
\partial_\eta^2 \theta + \frac{4}{\eta}\partial_\eta \theta - \frac{7}{3}\Delta\theta &= -\frac{1}{3}\partial_a\partial_b[\phi_{,a}\phi_{,b}]^t
\end{aligned}$$

with initial conditions $\phi(\eta_i, x) = \phi_i(x)$, $\partial_\eta\phi(\eta_i, x) = \phi_i'(x)$, $\theta(\eta_i, 0) = 0$, and $\partial_\eta\theta(\eta_i, x) = 0$. This is a semilinear hyperbolic initial value problem. In the Courant-Hilbert's book (Courant & Hilbert 1962) the unicity of solutions of this kind of problem is shown by means of the convergence of iterations. This supports the fact of considering the first iteration as criterion for validity of the linear approximation. In the following we shall focus on the reduced equation:

$$\partial_\eta^2 \phi + \frac{6}{\eta}\partial_\eta \phi - \frac{\tau^2}{\eta^4}\Delta\phi = -\frac{1}{6}(\nabla\phi)^2 \tag{46}$$

because it contains the essentials of the problem. Let us denote by $\phi^{(0)}$ the solution (33) to the linearized equation and by $\phi^{(1)}$ the first non linear correction, namely the solution of

$$\partial_\eta^2 \phi^{(1)} + \frac{6}{\eta}\partial_\eta \phi^{(1)} - \frac{\tau^2}{\eta^4}\Delta\phi^{(1)} = -\frac{1}{6}(\nabla\phi^{(0)})^2 \tag{47}$$

with null initial conditions. Using the Fourier transform method, the problem reduces to an ordinary differential equation:

$$\partial_\eta^2 \hat\phi^{(1)} + \frac{6}{\eta}\partial_\eta \hat\phi^{(1)} + \frac{\tau^2}{\eta^4} s\cdot s\hat\phi^{(1)} = L^{(0)}(\eta, k) \tag{48}$$

where $L^{(0)}(\eta, k)$ stands for the Fourier transform of the quadratic term $-\frac{1}{6}(\nabla\phi^{(0)})^2$, which is easily solved by the method of constants variation. The fundamental solutions $\{\hat\phi_1(\eta, s), \hat\phi_2(\eta, s)\}$ of the homogeneous equation were obtained in section 4, see expressions (28) and (29). Then, the solution of (48) can be expressed by means of integrals:

$$\hat\phi^{(1)}(\eta, k) = \frac{\eta_i}{\epsilon}\left(\hat\phi_2 \int_0^{g(\eta)} (1 - \frac{g}{\epsilon})^4 \hat\phi_1 L^{(0)} dg - \hat\phi_1 \int_0^{g(\eta)} (1 - \frac{g}{\epsilon})^4 \hat\phi_2 L^{(0)} dg\right)$$

and in the coordinate space it results:

$$\begin{aligned}
\phi^{(1)}(\eta, x) &= \frac{\eta_i}{\epsilon}\Bigg(Q_2 * \int_0^{g(\eta)} (1 - \tfrac{g}{\epsilon})^4 Q_1 * L^{(0)}(g, x) dg - \\
&- \quad Q_1 * \int_0^{g(\eta)} (1 - \tfrac{g}{\epsilon})^4 Q_2 * L^{(0)}(g, x) dg\Bigg)
\end{aligned} \tag{49}$$

being Q_1 and Q_2 the Green's functions given by (31) and (32). To decide about the validity of the linear approximation we need to compare $\mid \phi^{(1)}(\eta, x) \mid$ with $\mid \phi^{(0)}(\eta, x) \mid$. This is not easy to do directly, but using L^p norms we shall get an upper bound for the first non linear correction which will be enough to discuss the problem. Then, we have:

$$\mid \phi^{(1)}(\eta, x) \mid \ \leq \ \frac{\eta_i}{\epsilon}\left(\mid Q_2 * \int_0^{g(\eta)} (1-\frac{g}{\epsilon})^4 Q_1 * L^{(0)}(g,x) dg \mid + \ldots \right)$$

where dots means the same expression but interchanging Q_2 for Q_1. As before we are going to use the Hőlder inequalities, in particular $\parallel f * g \parallel_\infty \ \leq \ \parallel f \parallel_1 \ \parallel g \parallel_\infty$ and $\parallel f * g \parallel_2 \ \leq \ \parallel f \parallel_1 \ \parallel g \parallel_2$, where $\parallel g \parallel_2$ means $\int \mid g \mid^2 dx$. We get in this way:

$$\parallel \phi^{(1)}(\infty, x) \parallel_\infty \ \leq \ \frac{\eta_i}{\epsilon}\left(\parallel Q_2(\infty, x) \parallel_1 \int_0^{\epsilon} (1-\frac{g}{\epsilon})^4 \parallel Q_1(g,x) \parallel_\infty \parallel L^{(0)}(g,x) \parallel_1 dg + \ldots \right)$$

In the sequel we shall obtain this upper bound for the numerical example studied in subsection 5.2 corresponding to the initial conditions of the form $\phi_i(x) = 0$ and $\phi_i' \neq 0$. So, we write $\phi^{(0)}(\eta, x) = Q_2(\eta, x) * \phi_i'(x)$ and get the estimation $\parallel L^{(0)}(g,x) \parallel_{_1} \ \leq \ \frac{1}{6} \parallel Q_2(g,x) \parallel_{_1}^2 \sum_{a=1}^3 \parallel \nabla_a \phi_i'(x) \parallel_{_2}^2$. Substituting this into the previous equation we obtain:

$$\parallel \phi^{(1)}(\infty, x) \parallel_\infty \ \leq \ \frac{1}{6} G(\eta_i, \epsilon) \sum_{a=1}^{3} \parallel \nabla_a \phi_i'(x) \parallel_2^2 \tag{50}$$

with

$$\begin{aligned} G(\eta_i, \epsilon) \ &= \ \frac{\eta_i}{\epsilon}\Bigg(\parallel Q_2(\infty, x) \parallel_1 \int_0^{\epsilon} \left(1-\frac{g}{\epsilon}\right)^4 \parallel Q_1 \parallel_\infty \ \parallel Q_2 \parallel_1^2 dg + \\ &\qquad + \parallel Q_1(\infty, x) \parallel_1 \int_0^{\epsilon} \left(1-\frac{g}{\epsilon}\right)^4 \parallel Q_2 \parallel_\infty \ \parallel Q_2 \parallel_1^2 dg \Bigg) \end{aligned}$$

Evaluating the corresponding L^p norms we have:

$$G(\eta_i, \epsilon) = \frac{3\eta_i^4}{4\pi\epsilon^3}\left(\frac{1}{5}\int_0^1 (1-y)^4 (\frac{y^5}{5} + y^3 - y^4)^2 dy + \int_0^1 (1-y)^4 (\frac{y^5}{5} + y^3 - y^4)^2 (\frac{y^2}{2} + 1 - y) dy \right)$$

which reduces, once calculated the integrals, to:

$$G(\eta_i, \epsilon) = 10^{-5} \frac{27\eta_i^4}{4\pi\epsilon^3} \tag{51}$$

Then, in the case considered in the previous subsection 5.2, a simple calculation gives $\sum_{a=1}^3 \parallel \nabla_a \phi_i'(x) \parallel_2^2 \ = \ 3\pi^2 B^2 R^5$. Substituting these results into equation (50), we get that the upper bound $\mathcal{C}^{(1)}$ to the first non linear correction is:

$$\mid \phi^{(1)}(\infty, x) \mid \ \leq \ \parallel \phi^{(1)}(\infty, x) \parallel_\infty \ < \ \mathcal{C}^{(1)} = 10^{-4} \eta_i^4 \frac{R^3}{\epsilon^3} B^2 R^2 \tag{52}$$

Now, we have to compare this bound with the norm of the linear solution $\phi^{(0)}$. To do it we form the quotient $\Gamma = \mathcal{C}^{(1)} / \mid \phi^{(0)}(\infty, 0) \mid$. Given that for this example we have spherical symmetry, we can use the unidimensional integrals of Appendix B to calculate the modulus of $\phi^{(0)}$. Then, Γ expresses as a function of the initial characteristic length R, which is represented in Figure 6.

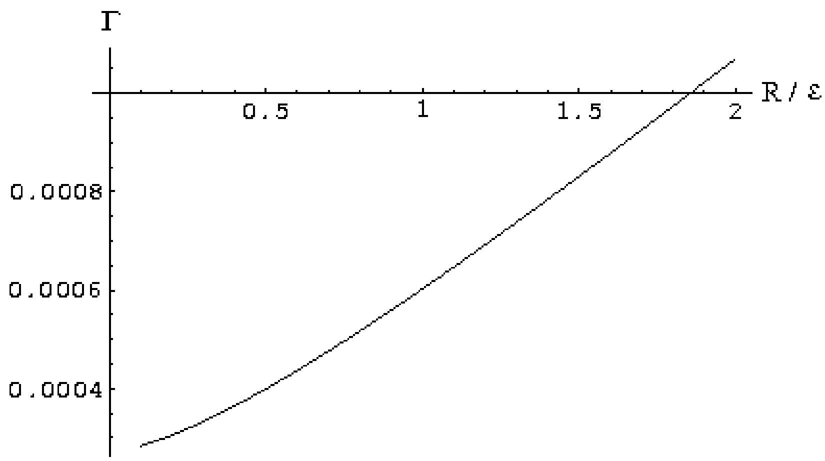

Fig. 6.— Validity of the linear aproximation of the evolution of the second degree of freedom showed in Figure 5. The function Γ is an upperbound, at the present epoch, of the ratio of the first non linear correction and the linear solution for the gravitational potential.

As we can see in this figure, the quotient Γ is below 10^{-2} in the range $R \leq 2\epsilon$, then the relation $\mid \phi^{(1)}(\infty, x) \mid < 0.01 \mid \phi^{(0)}(\infty, x) \mid$ is verified. So, as we have choosen $R = 1.6\epsilon$, we can neglect the first non linear correction and consequently, the linear approximation is an accurate description of this problem even having reached a great final density contrast.

Therefore, we can conclude that a thermal velocity τ of the order of 10^{-6} at the present epoch makes possible to follow with the linear approximation the formation of an inhomogeneity similar to a galaxy cluster. We have seen also that the linear approximation comes into problems with smaller values for the thermal velocity.

7. Conclusions

In the current theory on evolution of perturbations, the matter dominated epoch is considered as a fluid with null pressure, and the evolution is described using the linear

approximation until the density contrast becomes of the order of unity. However, as we have shown in this paper, if pressure is properly considered, the evolution with the linear approximation can be extended to values of the density contrast bigger than unity. We have assumed an isotropic pressure of the form $p = (a_o^2 T_o / m a^2(t))\rho$, which corresponds to an ideal gas with uniform temperature, and we have used reasonable values for the temperature. Concretely, in our examples we have taken a random mean square velocity of the order of $\sqrt{T_o/m} \approx 10^{-6}$.

In the following we summarize the main steps we have followed to get these conclusions:

1. We have stated a Cauchy problem using sistematically the 3+1 formalisme of General Relativity and neglecting quadratic terms in the metric perturbation. But, given that for inhomogeneities at scales of a few Mpc the spatial derivatives of the potential are much bigger than the potential, we have kept the quadratic terms in its first derivatives. These non linear corrections will only be used to study the validity of the linear approximation (see point 3). As usual, the coordinates are fixed by choosing the lapse function α and the shift vector β. We have put $\beta = 0$, and $\alpha = e^{b_1\phi + \phi^2}$ has been taken in order to simplify the evolution equations. When linearized, this choice of coordinates is called the longitudinal gauge.

2. We have obtained the solution of the linearized Cauchy problem for a one-component Universe in the matter dominated epoch and assuming an ideal gas equation of state $p = (a_o^2 T_o / m a^2(t))\rho$. We have expressed this solution in terms of convolution integrals of the initial conditions, which in our case are the initial potential $\phi_i(x)$ and its first time derivative $\phi_i'(x)$. We have also studied how to obtain a density inhomogeneity similar to a cluster of galaxies, i.e., how to get a total mass inside an Abel radius of the order of $3h^{-1} \times 10^{14}$ solar masses. We have considered separately both degrees of freedom obtaining in both cases an inhomogeneity similar to a galaxy cluster at the present epoch. But the initial density contrast and the initial macroscopic velocity in each case are very different:

   (a) With initial conditions of the type $\{\phi_i(x) \neq 0\ ,\ \phi_i'(x) = 0\}$, see subsection 5.1, one gets a cluster of galaxies starting at redshift $z_i = 5000$. These inital conditions correspond to an initial density contrast of $\delta_i \sim 0.1$ (which becomes about 0.5 at the recombination redshift), and a macroscopic velocity of $\mid V_i \mid \sim 3 \times 10^{-3}$. The results of this case can be also obtained with a Newtonian analysis starting with the same density contrast and macroscopic velocity.

   (b) With initial conditions of the type $\{\phi_i(x) = 0\ ,\ \phi_i'(x) \neq 0\}$, see subsectio 5.2, one gets a galaxy cluster starting at the same redshift $z_i = 5000$. But now one has a

very small initial density contrast $\delta_i \sim 0.0001$, and a similar macroscopic velocity, $\mid V_i \mid \sim 8 \times 10^{-3}$. Unlike the previous case, this evolution has no Newtonian analogue.

The second degree of freedom, which is currently forgotten, may rapidly produce inhomogeneities similar to galaxy clusters starting from faint initial density contrast. On the contrary, the first degree of freedom needs an excessive initial density contrast.

3. We have estimated the first non linear correction to the linear approximation and used it as the criterion of validity. Our first results seem quite interesting: assuming at the present epoch a thermal velocity of the order of $\tau \approx 10^{-6}$, the quotient between the first correction to the gravitational potential $\phi^{(1)}$ and the linear solution $\phi^{(0)}$ is small than 10^{-2}. Therefore, we can conclude that the linear approximation is an accurate description of the formation of a big structure if the effect of the pressure is not neglected.

A. Description of an ideal gas in the longitudinal gauge

An ideal gas in the matter dominated epoch with four-velocity w has a perfect fluid energy tensor $T_{\mu\nu} = \rho_c w_\mu w_\nu + p_c(g_{\mu\nu} + w_\mu w_\nu)$, with equation of state $p_c = T\rho_c/m$ being $T = T_o a_o^2/a^2$, where T_o is the temperature at the present epoch. The four-velocity w is related to the four-velocity $u = (1/\alpha)\partial_t$ by $w = \gamma(V)(u + V))$ where V is the macroscopic velocity of the Einstein-de Sitter perturbation in the longitudinal gauge, which is given by:

$$V_i = \frac{1}{4\pi G\rho}(-H\phi_{,i} - \frac{1}{a}\phi_{,\eta i} + \frac{1}{2a}\sigma_{im,m\eta} + H\sigma_{im,m})$$

where ρ is the energy density for the observer u. Using this relation, the energy tensor transforms into $T_{\mu\nu} = \rho u_\mu u_\nu + p(g_{\mu\nu} + u_\mu u_\nu) + q_\mu u_\nu + q_\nu u_\mu + \Pi_{\mu\nu}$, where now ρ, p and $\Pi_{\mu\nu}$ are quantities referred to the observer u and are given by:

$$\rho = \rho_c + O(V^2) \tag{A1}$$

$$p = p_c + \frac{1}{a^2}\rho V^2 + O(p_c V^2) \tag{A2}$$

$$\Pi_{ij} = \rho[V_i V_j]^t + O(\rho V^4) \tag{A3}$$

Given the equation of state and the relation $\rho_c = \rho_{_B}(1+\delta)$ we can write $p_c = T\rho_{_B}/m + \delta\rho_{_B} T/m$, which allows us to get:

$$p = \frac{T}{m}\rho_{_B} + \frac{\tau^2}{4\pi G a^2}\left(\frac{1}{a^2}\Delta\phi - \frac{3H}{a}\phi_\eta - 3H^2\phi\right) +$$

$$+\frac{1}{18\pi G a^2 H^2(1+\delta)}\left(H^2(\nabla\phi)^2+\frac{1}{a^2}(\nabla\phi')^2+2\frac{H}{a}\nabla\phi\cdot\nabla\phi'\right)$$

where we have also introduced the r.m.s. velocity at the present epoch $\tau^2 = (T_o/m)$. From this expression and comparing with equation (7), we identify:

$$E = \frac{\tau^2}{4\pi G a^4} \quad \text{(A4)}$$

$$p^{(2)} = \frac{1}{18\pi G a^2 H^2(1+\delta)}\left(H^2(\nabla\phi)^2+\frac{1}{a^2}(\nabla\phi')^2+2\frac{H}{a}\nabla\phi\cdot\nabla\phi'\right) \quad \text{(A5)}$$

As for the anisotropic pressures, in the same way, equation (A3) gives:

$$\Pi_{ij} = \frac{1}{6\pi G H^2(1+\delta)}\left(H^2[\phi_i\phi_j]^t+\frac{1}{a^2}[\phi'_i\phi'_j]^t+\frac{H}{a}[\phi'_i\phi_j+\phi_i\phi'_j]^t\right) \quad \text{(A6)}$$

and comparing with (6) we obtain $A=0$, and $\pi^{(2)}_{ij} = \Pi_{ij}$.

Let us write the non linear evolution equations (10) and (11) as follows:

$$\begin{aligned}\mathcal{L}^1(\phi,\sigma) &= \frac{1}{2}(\partial_\eta\phi)^2-\frac{1}{6}(\nabla\phi)^2+4\pi G a^2 p^{(2)}\\ \mathcal{L}^2(\sigma) &= 4(1-16\pi G A)[\phi_{,i}\phi_{,j}]^t+16\pi G(1-8\pi G A)\pi^{(2)}_{ij}\end{aligned}$$

where we have used a compact notation for the first members of the equations. These expressions, substituting A, $p^{(2)}$ and $\pi^{(2)}_{ij}$, transform in:

$$\mathcal{L}^1(\phi,\sigma) = \frac{1}{2}(\partial_\eta\phi)^2-\frac{1}{6}(\nabla\phi)^2+\frac{2}{3H^2(1+\delta)}\left(H^2(\nabla\phi)^2+\frac{1}{a^2}(\nabla\phi')^2+2\frac{H}{a}\nabla\phi\cdot\nabla\phi'\right) \quad \text{(A7)}$$

$$\mathcal{L}^2(\sigma) = 4[\phi_{,i}\phi_{,j}]^t+\frac{8}{3H^2(1+\delta)}\left(H^2[\phi_i\phi_j]^t+\frac{1}{a^2}[\phi'_i\phi'_j]^t+\frac{H}{a}[\phi'_i\phi_j+\phi_i\phi'_j]^t\right) \quad \text{(A8)}$$

And taking into account that the density contrast δ will have a great value in the structures we are interested on, the terms where it appears can be neglected.

Let us to point out that keeping these terms only would produce small corrections to the upper bounds estimations for the first nonlinear correction to the linear approximation as obtained in section 6.

B. Obtaining the Green functions

In this appendix we are going to summarize the process to obtain the Green's functions associated to the general solution of our initial value Cauchy problem (26). This general

solution in the Fourier space has the form:

$$\hat{\phi}(\eta, s) = \hat{\phi}_i(s)\hat{\phi}_1(\eta, s) + \hat{\phi}_i'(s)\hat{\phi}_2(\eta, s).$$

being $\hat{\phi}_i(s)$ and $\hat{\phi}_i'(s)$ the Fourier transform of the initial conditions and $\hat{\phi}_1(\eta, s)$ and $\hat{\phi}_2(\eta, s)$ the fundamental solutions given by (28) and (29). In order to have the general solution in the coordinate space we need to make the corresponding inverse Fourier transforms with respect to the spatial coordinates. Taking into account that, in this general development, the initial conditions are arbitrary functions of the spatial coordinates, the best way to give this general solution will be using the convolution product between functions (denoted by $*$) with respect to the spatial coordinates. That is, the properties of the inverse Fourier transforms and of the convolution product allow us to write the solution as:

$$\phi(\eta, x) = Q_1(\eta, x) * \phi_i(x) + Q_2(\eta, x) * \phi_i'(x)$$

where $Q_1(\eta, x)$ and $Q_2(\eta, x)$ are the Green functions, that is the inverse Fourier transforms of the fundamental solutions $\hat{\phi}_1(\eta, s)$ and $\hat{\phi}_2'(\eta, s)$ respectively. Then, we need to calculate a few inverse Fourier transforms to obtain the Green functions.

Firstly, we consider that the fundamental functions (28) and (29) can also be written in the following form:

$$\hat{\phi}_1(\eta, s) = \frac{3}{\epsilon^3}\mathcal{D}_g\left(\frac{\sin kg}{k^3}\right) + \frac{\eta_i(3\eta - \eta_i)}{\epsilon\eta^2}\frac{\sin kg}{k} + \frac{\eta_i^2}{\eta^2}\partial_g\left(\frac{\sin kg}{k}\right) \tag{B1}$$

$$\hat{\phi}_2(\eta, s) = \frac{9\eta_i}{\epsilon^5}\left(\mathcal{D}_g + \frac{1}{3}g^2\partial_g^2\right)\frac{\sin kg}{k^5} + \frac{3\eta_i^2}{\epsilon^3\eta}\mathcal{D}_g\left(\frac{\sin kg}{k^3}\right) + \frac{\eta_i^3}{\epsilon\eta^2}\frac{\sin kg}{k} \tag{B2}$$

where $g = \epsilon\left(1 - \frac{\eta_i}{\eta}\right)$, $\epsilon = \tau/\eta_i$, $k = \sqrt{s \cdot s}$ and being the operator $D_g(f) = f - g\partial_g f$. This form simplifies the number of inverse Fourier tranforms that we have to obtain. In fact, we shall only need a pair of well–known inverse Fourier transforms as we shall see below.

Let us remind the definition of the inverse Fourier transform of a function with respect to the spatial coordinates, that is:

$$\mathcal{F}^{-1}[W(s, \eta)] = \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3} e^{-is\cdot x}W(s, \eta)ds.$$

With this definition, we have that the inverse Fourier transform of the function $(\sin kg)/k$ is known and has the following general form:

$$\mathcal{F}^{-1}\left[\frac{\sin k\lambda}{k}\right] = \frac{1}{4\pi\lambda}\left\{\delta_D(\lambda - r)H(\lambda) + \delta_D(\lambda + r)H(-\lambda)\right\} \tag{B3}$$

where $\delta_D(x)$ represents the Dirac delta distribution and $H(\lambda)$ is the Heaviside unity function. On the other hand, we also have the general expression (Guelfand & Chilov 1962):

$$\mathcal{F}^{-1}[k^{-\lambda-n}] = \frac{\Gamma(\frac{-\lambda}{2})r^{\lambda}}{2^{\lambda+n}\Gamma(\frac{\lambda+n}{2})\pi^{3/2}} \tag{B4}$$

where n denotes the dimension of the space where are realized the inverse Fourier transforms and $\Gamma(\cdot)$ represents the Gamma function. These two expressions will allow us to calculate all the inverse Fourier transforms involved in the Green's functions.

Having a look to the fundamental solutions (B1) and (B2) we can see that we need the inverse Fourier tranforms of functions of the form $(\sin kg)/k^p$, which can be obtained from (B3) and (B4) using the convolution product in the following way:

$$\mathcal{F}^{-1}\left[\frac{\sin k\lambda}{k^p}\right] = \mathcal{F}^{-1}\left[\frac{\sin k\lambda}{k}\right] * \mathcal{F}^{-1}\left[k^{-(p-1)}\right] .$$

As the convolution product in general is given by the expression:

$$(f * g)(x) = \int_{\mathbb{R}^n} f(t)g(x-t)dt \tag{B5}$$

it results that the two inverse Fourier tranforms needed are expressed in general as:

$$\mathcal{F}^{-1}\left[\frac{\sin k\lambda}{k^3}\right] = \frac{1}{4\pi}\left\{\left[H(\lambda-r)+\frac{\lambda}{r}H(r-\lambda)\right]H(\lambda)-\left[H(-\lambda-r)-\frac{\lambda}{r}H(\lambda+r)\right]H(-\lambda)\right\}$$

$$\begin{aligned}\mathcal{F}^{-1}\left[\frac{\sin k\lambda}{k^5}\right] = \frac{-1}{24\pi r}\Big\{ &\left[r(r^2+3\lambda^2)H(\lambda-r)+\lambda(\lambda^2+3r^2)H(r-\lambda)\right]H(\lambda)- \\ &-\left[r(r^2+3\lambda^2)H(-\lambda-r)-\lambda(\lambda^2+3r^2)H(\lambda+r)\right]H(-\lambda)\Big\}\end{aligned}$$

The corresponding operators acting over these expressions will give us that the Green functions have the form:

$$Q_1(\eta,x) = \frac{3}{4\pi\epsilon^3}H(g-r)+\frac{\eta_i(3\eta-\eta_i)}{4\pi\epsilon\eta^2}\frac{\delta_D(r-g)}{g}+\frac{\eta_i^2}{4\pi\eta^2}\partial_g\left(\frac{\delta_D(r-g)}{g}\right)$$

$$Q_2(\eta,x) = \left(\frac{3\eta_i}{8\pi\epsilon^5}(g^2-r^2)+\frac{3\eta_i^2}{4\pi\epsilon^3\eta}\right)H(g-r)+\frac{\eta_i^3}{4\pi\epsilon\eta^2}\frac{\delta_D(r-g)}{g}$$

To obtain these expressions we have considered that g is always a positive number (that is, $H(g)=1$ and $H(-g)=0$).

Finally, as we have said above, the general solution of the initial value Cauchy problem in the real space is given by the convolution product between the Green functions and the initial conditions, that is:

$$\phi(\eta, x) = Q_1(\eta, x) * \phi_i(x) + Q_2(\eta, x) * \phi_i'(x).$$

This expression reduces to unidimensional integrals in the case of spherical symmetry. That is, if we consider the initial conditions $\phi_i(x) = f_1(r)$ and $\phi_i'(x) = f_2(r)$ as functions depending only on the radial coordinate r, then the corresponding convolution product, defined by (B5), is written as unidimensional integrals as we show in the following.

To make clear the expressions we can consider firstly the case when $f_1(r) \neq 0$ and $f_2(r) = 0$. In this case the final gravitational potential will be written as:

$$\phi(\eta, r) = \frac{3}{2\epsilon^3}\left\{2\int_0^{g-r} q^2 f_1(q)\, dq\; H(g-r) + \frac{1}{2r}\int_{|r-g|}^{r+g} q f_1(q)\Big(2rq + K(r,q,g)\Big)dq\right\} +$$

$$+\frac{\eta_i g(3\eta - \eta_i)}{2\eta^2\epsilon}\int_{-1}^{1} f_1(\sqrt{r^2+g^2-2rgx})dx + \frac{\eta_i^2}{2\eta^2}\int_{-1}^{1} f_1(\sqrt{r^2+g^2-2rgx})dx +$$

$$+\frac{g\eta_i^2}{2\eta^2}\int_{-1}^{1}\frac{g - rx}{\sqrt{r^2+g^2-2rgx}} f_1'(\sqrt{r^2+g^2-2rgx})dx$$

where $K(r,q,g) = g^2 - r^2 - q^2$.

On the other hand, the case in which we have $f_1(r) = 0$ and $f_2(r) \neq 0$ we will have that the final evolution of $\phi(\eta, x)$ is:

$$\phi(\eta, r) = \frac{3\eta_i}{4\epsilon^5}\left\{2\int_0^{g-r} q^2 f_2(q) K(r,q,g) dq H(g-r) - \frac{1}{4r}\int_{|r-g|}^{r+g} q f_2(q)\Big(K(r,q,g)^2 + 4r^2q^2\Big)dq +\right.$$

$$\left.+\frac{1}{2r}\int_{|r-g|}^{r+g} qK(r,q,g) f_2(q)\left(2rq + K(r,q,g)\right)dq\right\} + \frac{\eta_i^3 g}{2\eta^2\epsilon}\int_{-1}^{1} f_2(\sqrt{r^2+g^2-2rgx})dx +$$

$$+\frac{3\eta_i^2}{2\eta\epsilon^2}\left\{2\int_0^{g-r} q^2 f_2(q) dq H(g-r) + \frac{1}{2r}\int_{|r-g|}^{r+g} q f_2(q)\left(2rq + K(r,q,g)\right)dq\right\}$$

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